THE CUP SUBALGEBRA OF A II_1 FACTOR GIVEN BY A SUBFACTOR PLANAR ALGEBRA IS MAXIMAL AMENABLE

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ABSTRACT. To every subfactor planar algebra was associated a II_1 factor with a canonical abelian subalgebra generated by the cup tangle. Using Popa's approximative orthogonality property, we show that this cup subalgebra is maximal amenable.

INTRODUCTION

The study of maximal abelian subalgebras (MASAs) has been initiated by Dixmier [Dix54] where he introduced an invariant coming from the normalizer. Other invariants have been provided later, like the Takesaki equivalence relation [Tak63], the length of Tauer [Tau65], the Pukanszky invariant [Puk60] or the δ-invariant [Pop83b].

Popa exhibits in [Pop83a] an example of a MASA $A \subset M$ in a II_1 factor that is maximal amenable.

This example answers negatively to a question of Kadison that asks if every abelian subalgebra of a II_1 factor (with separable predual) is included in a copy of the hyperfinite II_1 factor. We recall that a von Neumann algebra is hyperfinite if and only if it is amenable by the famous theorem of Connes [Con76]. Popa introduced the notion of approximative orthogonality property (in short AOP) and showed that any singular MASA with the AOP is maximal amenable. Then he proved that the generator MASA in a free group factor is singular and has the AOP.

Using the same scheme of proof, Cameron et al. [CFRW10] showed that the radial MASA in the free group factor is maximal amenable. Also Shen [She06], Jolissaint [Jol10] and Houdayer [Hou12] provided other examples of maximal amenable MASAs.

In this paper, we provide maximal amenable MASAs in II_1 factors using subfactor planar algebras. The theory of subfactors has been initiated by Jones [Jon83]. He introduced the standard invariant that has been formalized as a Popa system by Popa [Pop95] and as a subfactor planar algebra by Jones [Jon99, Jon12]. Popa [Pop93, Pop95, Pop02] proved that any standard invariant comes from a subfactor. Popa and Shlyakhtenko proved [PS03] that the subfactor can be realized in the infinite free group factor $L(\mathbb{F}_\infty)$. Using planar algebras, random matrix models and free probability, Guionnet et al. [GJS10, JSW10, GJS11] showed that any finite depth standard invariant can be realized as a subfactor of an interpolated free group factor. Using the same construction, Hartglass [Har12] proved that any infinite depth subfactor is realized in $L\mathbb{F}_\infty$.

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The construction of Jones et al. [JSW10] associated a II_1 factor M to a subfactor planar algebra $\mathcal{P}$. This II_1 factor contains a generic MASA $A \subset M$, see section 2.2, that we call the *cup subalgebra*. The main theorem of this paper is

THEOREM 0.1. *For any non trivial subfactor planar algebra $\mathcal{P}$, the cup subalgebra is maximal amenable.*

Note that the construction of [JSW10] has been extended for unshaded planar algebras in [Bro12] and in [BHP12]. In those construction, we have proven that the cup subalgebra is still a MASA. It seems very plausible that it is also maximal amenable. Note that the cup subalgebra is analogous of the *radial MASA* in a free group factor. We don't know if for a certain subfactor planar algebra those tow subalgebras are isomorphic or not.

1. APPROXIMATIVE ORTHOGONALITY PROPERTY AND MAXIMAL AMENABILITY

We briefly recall Popa's approximative orthogonality property for an abelian subalgebra $A \subset M$ and how it implies the maximal amenability of A, whenever $A \subset M$ is a singular MASA.

DEFINITION 1.1. (see [Pop83a, Lemma 2.1]) Consider a tracial von Neumann algebra (M, tr) and a subalgebra $A \subset M$. Let ω be a free ultrafilter on $\mathbb{N}$. Then $A \subset M$ has the approximative orthogonality property (in short AOP) if for any $x \in M^\omega \ominus A^\omega \cap A'$ and any $b \in M \ominus A$ we have $xb \perp bx$ in $L^2(M^\omega)$, i.e. $\lim_{n\to\omega} tr(x_n b x_n^* b^*) = 0$ where $(x_n)_n$ is a representative of x.

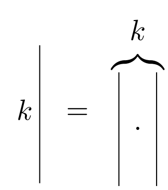
REMARK 1.2. By polarization, the definition of AOP is equivalent to ask that for any $x_1, x_2 \in M^\omega \ominus A^\omega \cap A'$ and any $b_1, b_2 \in M \ominus A$ we have $x_1 b_1 \perp b_2 x_2$.

We recall the fundamental theorem of Popa that is contained in the proof of [Pop83a, Theorem 3.2]. A more detailed explanation of Popa's theorem has been given in [CFRW10, Lemma 2.2 and Corollary 2.3].

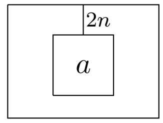
THEOREM 1.3. [Pop83a] *Let $A \subset M$ be a singular MASA with the AOP in a II_1 factor M. Then $A \subset M$ is maximal amenable.*

2. CONSTRUCTION OF THE CUP SUBALGEBRA

2.1. **Construction of a II_1 factor from a subfactor planar algebra.** Consider a subfactor planar algebra $\mathcal{P} = (\mathcal{P}_n)_{n\geqslant 0}$ with modulus $\delta > 1$. Let us recall the construction given in [JSW10]. We assume that the reader is familiar with planar algebras. For more details on planar algebras, see Jones [Jon99, Jon12] or the introduction of Peters [Pet10]. Let $Gr(\mathcal{P})$ be the graded vector space equal to the algebraic direct sum $\bigoplus_{n\geqslant 0} \mathcal{P}_n$. We decorate strands in a planar tangle with natural numbers to represent cabling of that strand. For example



An element $a \in \mathcal{P}_n$ will be represent as a box:

$a =$ 

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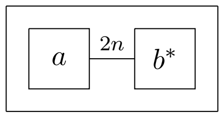
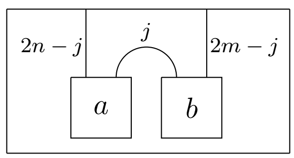
We assume that the distinguished first interval is at the top left of the box. We consider the inner product $\langle \cdot, \cdot \rangle$ on each $\mathcal{P}_n$ that is:

$\langle a, b \rangle =$ 

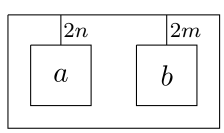
, for all $a, b \in \mathcal{P}_n$.

We extend this inner product on $Gr(\mathcal{P})$ in such a way that the spaces $\mathcal{P}_n$ are pairwise orthogonal. We still write $\mathcal{P}_n$ when it is considered as the n-graded part of $Gr(\mathcal{P})$. Let $\mathcal{H}$ be the Hilbert space equal to the completion of $Gr(\mathcal{P})$ for its prehilbert structure. Note that $\mathcal{H}$ is the Hilbert space equal to the orthogonal direct sum of the spaces $\mathcal{P}_n$. We define a multiplication on $Gr(\mathcal{P})$ given by the tangle:

$ab = \sum_{j=0}^{\min(2n,2m)}$ 

, for all $a \in \mathcal{P}_n$, $b \in \mathcal{P}_m$.

For a fix $a \in Gr(\mathcal{P})$, the map $b \in Gr(\mathcal{P}) \longmapsto ab \in Gr(\mathcal{P})$ is bounded for the inner product $\langle \cdot, \cdot \rangle$. This gives us a representation of the $*$-algebra $Gr(\mathcal{P})$ on $\mathcal{H}$. We denote by M the von Neumann algebra equal to the bicommutant of this representation. It is a II_1 factor by [JSW10]. We identify the graded algebra $Gr(\mathcal{P})$ and its image in the von Neumann algebra M. The unique faithful normal trace tr of M is the one coming from the planar algebra structure of $\mathcal{P}$. It is equal to the formula $tr(a) = \langle a, 1 \rangle$, where 1 is the unity of $Gr(\mathcal{P})$. Let $L^2(M)$ be the Hilbert space coming from the GNS construction over the trace tr. Note that the standard representation of the von Neumann algebra M on the Hilbert space $L^2(M)$ is conjugate to the action of M on the Hilbert space $\mathcal{H}$. We will identify those two representations. Also, we identify M with its image in $L^2(M)$. The left and right action of M in the GNS construction $L^2(M)$ are denote by π and ρ, i.e. $\pi(x)\rho(y)z = xzy$, for $x, y, z \in M$. If a confusion is possible, we denote by $\| \cdot \|_2$ the norm of $L^2(M)$ and by $\| \cdot \|$ the norm of M. We define a multiplication on $Gr(\mathcal{P})$ by requiring that if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$, then $a \bullet b \in \mathcal{P}_{n+m}$ is given by

$a \bullet b =$ 

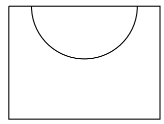
.

We remark that $\|a \bullet b\|_2 = \|a\|_2 \|b\|_2$, for all $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. By the triangle inequality, the bilinear function

$$\begin{aligned} Gr(\mathcal{P}) \times Gr(\mathcal{P}) &\longrightarrow Gr(\mathcal{P}) \\ (a, b) &\longmapsto a \bullet b \end{aligned}$$

is continuous for the norm $\| \cdot \|_2$. We extend this operation on $L^2(M) \times L^2(M)$ and still denote it by $\bullet$.

2.2. **The cup subalgebra.** The cup subalgebra $A \subset M$ is the abelian von Neumann algebra generated by the self-adjoint element cup: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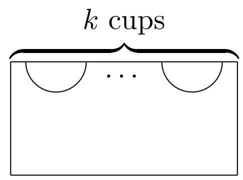
We denote cup by the symbol $\cup$. Also we use the following notation

$\cup^{\bullet k} =$


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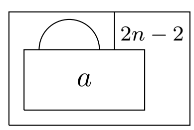
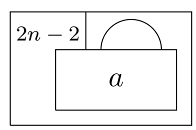
We use the convention that $0 = \cup^{\bullet k}$ for $k < 0$ and $1 = \cup^{\bullet 0}$. Let $n \geqslant 1$ and V_n be the subspace of $\mathcal{P}_n$ of elements which vanish when a cap is placed at the top right and vanish when a cap is placed at the top left, i.e.

$V_n = \Big\{ a \in \mathcal{P}_n,$


$=$


$= 0 \Big\}$.

We denote by V the orthogonal direct sum of the V_n, i.e.

$$V = \bigoplus_{n=1}^{\infty} V_n.$$

Let $\ell^2(\mathbb{N})$ be the separable Hilbert space with orthonormal basis $\{e_n,\ n \geqslant 0\}$ and $S \in \mathbb{B}(\ell^2(\mathbb{N}))$ the unilateral shift operator.

Proposition 2.1. *The map*

$$\begin{array}{rccccc} \Theta : & L^2(M) & \longrightarrow & \ell^2(\mathbb{N}) & \oplus & (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})) \\ & \delta^{-\frac{k}{2}} \cup^{\bullet k} & \longmapsto & e_k & \oplus & 0 \\ & \delta^{-\frac{l+r}{2}} \cup^{\bullet l} \bullet v \bullet \cup^{\bullet r} & \longmapsto & 0 & \oplus & e_l \otimes v \otimes e_r \end{array}$$

defines a unitary transformation, where $k, l, r \geqslant 0$ and $v \in V$. We have that

$$\Theta\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\Theta^* = \begin{pmatrix} S + S^* - q_{e_0} & 0 \\ 0 & (S + S^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} \end{pmatrix}$$

and

$$\Theta\rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\Theta^* = \begin{pmatrix} S + S^* - q_{e_0} & 0 \\ 0 & 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (S + S^*) \end{pmatrix},$$

where q_{e_0} is the rank one projection on $\mathbb{C}e_0$.

Proof. See [JSW10][theorem 4.9.]. □

Corollary 2.2. *The cup subalgebra is a singular MASA.*

PROOF. The A-bimodule $L^2(M) \ominus L^2(A)$ is isomorphic to an infinite direct sum of the coarse bimodule $L^2(A) \otimes L^2(A)$. This implies that $A \subset M$ is maximal abelian. See [JSW10] for more details. Suppose that there exists u in the normalizer of A inside M which is orthogonal to A. It generates a A-subbimodule

$$\mathcal{K} \subset \bigoplus_{j=0}^{\infty} L^2(A) \otimes L^2(A). \tag{1}$$

We have the inclusion 1 if and only if the automorphism $a \in A \mapsto uau^*$ is trivial. This implies that $u \in A' \cap M$. Hence $u \in A$, a contradiction. Therefore, $A \subset M$ is singular. □

2.3. **Basic facts on the unilateral shift operator.** Consider the semi-circular measure

$$d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi} dt$$

defined on the interval $[-2;2]$. Let $P_i \in \mathbb{R}[X]$ be the family of polynomials such that $P_0(X) = 1$, $P_1(X) = X$ and $P_i(X) = XP_{i-1}(X) - P_{i-2}(X)$ for all $i \geqslant 2$. By [DNV92, example 3.4.2], we have that the map

$$\Psi : \ell^2(\mathbb{N}) \longrightarrow L^2([-2;2], \nu)$$
$$e_i \longmapsto P_i$$

defines a unitary transformation. Furthermore, for any continuous function $f \in \mathcal{C}([-2;2])$, $(\Psi^* f(S+S^*)\Psi)(t) = tf(t)$ for almost every $t \in [-2;2]$.

LEMMA 2.3. *Consider $I \geqslant 0$ and the function $R_I : [-2;2] \longrightarrow \mathbb{R}$ such that $R_I(t) = \sum_{i=0}^{I} P_i(t)^2$. The sequence $(R_I)_{I \geqslant 0}$ converges uniformly to $+\infty$.*

PROOF. Let us prove the simple convergence to $+\infty$. Suppose there exists $t_0 \in [-2;2]$ such that the sequence $(R_I(t_0))_k$ does not converge to $+\infty$. The polynomial P_i have real coefficient, hence for any $t \in [-2;2]$, $P_i(t)$ is real; thus, $(R_I(t_0))_k$ is an increasing sequence in $\mathbb{R}$. If this sequence does not diverge, then it is bounded. Then, the sequence $(P_i(t_0))_i$ is square summable. In particular we have $\lim_{i\to\infty} P_i(t_0) = 0$. We put $\varepsilon_i = P_i(t_0)$. We have that $\varepsilon_{i+1} = t_0\varepsilon_i - \varepsilon_{i-1}$ and $\lim_{i\to\infty} \varepsilon_i = 0$. There is only one sequence that satisfies those axioms and it is the sequence equal to zero. Since $0 \neq 1 = P_0(t_0) = \varepsilon_0$, we arrive to a contradiction and thus, $\lim_{I\to\infty} S_I(t) = +\infty$ for any $t \in [-2;2]$. To conclude we use the following well known result due to Dini: Let $(f_I)_I$ be a sequence of continuous functions from a compact topological space K to $\mathbb{R}$ such that $f_I \leqslant f_{I+1}$. If for any $t \in K$, $\lim_{I\to\infty} f_I(t) = +\infty$, then the sequence $(f_I)_I$ converges uniformly to $+\infty$. □

2.4. **Proof of Theorem 0.1.** According to the Theorem 1.3 and Corollary 2.2 it is sufficient to show that the cup subalgebra has the AOP. Fix $x \in M^\omega \ominus A^\omega \cap A'$ and $b \in M \ominus A$. Let us show that $xb \perp bx$. By the density theorem of Kaplansky, we can assume that their exists $J \geqslant 1$ such that $b \in \bigoplus_{j=0}^{J} \mathcal{P}_j$. Suppose that $\|x\|, \|b\| \leqslant 1$. Fix a sequence $x_n \in M$ which is a representative of x such that $x_n \in M \ominus A$ and $\|x_n\| \leqslant 1$ for all $n \geqslant 0$.

Consider the closed subspaces of $L^2(M)$:

$$Y_L = \overline{\text{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r},\ l, r \leqslant L,\ v \in V\} \text{ and}$$
$$Z_L = \overline{\text{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r},\ l \leqslant L,\ v \in V, r \geqslant 0\},$$

for all $L \geqslant 0$. Remark that b is in Y_{J-1}.

We claim that

$$zb \perp bx, \text{ for all } z \in M \cap L^2(M) \ominus L^2(A) \ominus Z_{J-1}. \tag{2}$$

The element z is a weak limit of finite linear combinations of $\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j}$, where $i \geqslant J$, $v \in V$ and $j \geqslant 0$. The element b is a finite linear combination of $\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r}$, where $k, r \geqslant 0$ and $\tilde{v} \in V$. We have

$$\begin{aligned}(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j})(\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r}) = (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k} \bullet \tilde{v} \bullet \cup^{\bullet r}) + (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k-1} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \cdots \\ + \delta^j(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet k-j} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \delta^j(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet k-j-1} \bullet \tilde{v} \bullet \cup^{\bullet r}).\end{aligned}$$

It is easy to see that $v \bullet \cup^{\bullet n} \bullet \tilde{v}$ is an element of V for any n. Hence, the product $(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j})(\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r})$ is in the vector space

$$\overline{\text{span}}\{\cup^{\bullet l} \bullet w \bullet \cup^{\bullet r},\ l \geqslant J,\ w \in V,\ r \geqslant 0\} = L^2(M) \ominus L^2(A) \ominus Z_{J-1}$$

Therefore, zb is also in $L^2(M) \ominus L^2(A) \ominus Z_{J-1}$. A similar computation shows that bx is in

$$\overline{\text{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r},\ l \leqslant J-1,\ w \in V,\ r \geqslant 0\} = Z_{J-1}$$

Therefore, we have $zb \perp bx$. This proves 2. Consider Q_J, the orthogonal projection of range Z_{J-1}. By 2 we have

$$\begin{aligned}|\langle xb, bx\rangle| &= \lim_{n\to\omega} |\langle Q_J(x_n)b, bx_n\rangle| \\ &\leqslant \lim_{n\to\omega} \|Q_J(x_n)\|_2 \|b\| \|bx_n\|_2 \leqslant \lim_{n\to\omega} \|Q_J(x_n)\|_2.\end{aligned}$$

Let us show that $\lim_{n\to\omega} \|Q_J(x_n)\|_2 = 0$. Consider the unitary transformation Θ of Section 2.3. We put $\xi_n = \Theta x_n$. Remark that

$$\Theta Q_J \Theta^* = 1_{\ell^2(\mathbb{N})} \oplus \bigoplus_{i=0}^{J-1} q_{e_i} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})}.$$

Also the vector ξ_n belongs to $\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})$. Therefore it is sufficient to show that

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})})\xi_n\| = 0, \text{ for any } i \geqslant 0. \tag{3}$$

The element x commutes with A. Hence by conjugation by Θ we have

$$\lim_{n\to\omega} \|((S+S^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} - 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (S+S^*))\xi_n\| = 0. \tag{4}$$

We will show that 4 implies 3.

All the operators involved in our context act trivially on the factor V. For simplicity of the notations we stop writing the extra "$\otimes 1_V \otimes$" in the formula. We denote $1_{\ell^2(\mathbb{N})}$ by 1. Therefore, we assume that ξ_n is a vector of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. The Equation 3 and 4 become

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1)\xi_n\| = 0, \text{ for any } i \geqslant 0 \text{ and} \tag{5}$$

$$\lim_{n\to\omega} \|((S+S^*) \otimes 1 - 1 \otimes (S+S^*))\xi_n\| = 0. \tag{6}$$

Consider the partial isometry $v_i \in \mathbb{B}(\ell^2(\mathbb{N}))$ such that $v_i^* v_i = q_{e_i}$ and $v_i v_i^* = q_{e_0}$. We claim that for all $i \geqslant 0$ we have

$$\lim_{n\to\omega} \|((v_i \otimes 1) - (q_{e_0} \otimes P_i(S+S^*)))\xi_n\| = 0, \tag{7}$$

where $\{P_i\}_i$ is the family of polynomials defined in Section 2.3. We remark that for all $k \geqslant 0$ we have

$$(S+S^*)^k \otimes 1 - 1 \otimes (S+S^*)^k = ((S+S^*) \otimes 1 - 1 \otimes (S+S^*)) \circ (\sum_{j=0}^{k-1} (S+S^*)^j \otimes (S+S^*)^{k-1-j}).$$

Therefore, the Equation 6 implies that

$$\lim_{n\to\omega} \|(P(S+S^*) \otimes 1 - 1 \otimes P(S+S^*))\xi_n\| = 0, \text{ for all polynomials } P.$$

In particular,

$$\lim_{n\to\omega} \|(P_i(S+S^*) \otimes 1 - 1 \otimes P_i(S+S^*))\xi_n\| = 0, \text{ for all } i \geqslant 0.$$

Note that

$$P_i(S+S^*)(e_0) = e_i, \text{ for all } i \geqslant 0.$$

Furthermore, P_i has real coefficient. Therefore, the operator $P_i(S+S^*)$ is self-adjoint. We have

$$\begin{aligned}\langle q_{e_0} \circ P_i(S+S^*)e_l, e_r\rangle &= \langle P_i(S+S^*)e_l, q_{e_0} e_r\rangle = \delta_{r,0}\langle P_i(S+S^*)e_l, e_0\rangle \\ &= \delta_{r,0}\langle e_l, P_i(S+S^*)e_0\rangle = \delta_{r,0}\delta_{l,i},\end{aligned}$$

where $i, l, r \geqslant 0$ and $\delta_{n,m}$ is the Kronecker symbol. This shows that $q_{e_0} \circ P_i(S+S^*) = v_i$, for all $i \geqslant 0$. We have

$$\lim_{n\to\omega} \|(q_{e_0} \otimes 1) \circ (P_i(S+S^*) \otimes 1 - 1 \otimes P_i(S+S^*))\xi_n\| = 0.$$

Therefore, we have

$$\lim_{n\to\omega} \|(v_i \otimes 1 - q_{e_0} \otimes P_i(S+S^*))\xi_n\| = 0.$$

This proves the claim.

This implies that

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1 - v_i^* q_{e_0} \otimes P_i(S+S^*))\xi_n\| = 0.$$

This means that

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1)\xi_n - (v_i^* \otimes P_i(S+S^*)) \circ (q_{e_0} \otimes 1)\xi_n\| = 0.$$

Hence, we have

$$\begin{aligned}\lim_{n\to\omega} \|(q_{e_i} \otimes 1)\xi_n\| &\leqslant \lim_{n\to\omega} \|(v_i^* \otimes P_i(S+S^*)) \circ (q_{e_0} \otimes 1)\xi_n\| \\ &\leqslant \|v_i^* \otimes P_i(S+S^*)\| \lim_{n\to\omega} \|(q_{e_0} \otimes 1)\xi_n\|.\end{aligned}$$

Therefore, to prove 5 it is sufficient to show that

$$\lim_{n\to\omega} \|(q_{e_0} \otimes 1)\xi_n\| = 0.$$

Let us fix $\varepsilon > 0$, we have to find an element of the ultrafilter $E \in \omega$ such that for any $n \in E$, $\|(q_{e_0} \otimes 1)\xi_n\| < \varepsilon$. By the triangle inequality, we have

$$\|(q_{e_0} \otimes P_i(S+S^*))\xi_n\| \leqslant \|(q_{e_0} \otimes P_i(S+S^*))\xi_n - (v_i \otimes 1)\xi_n\| + \|(v_i \otimes 1)\xi_n\|,$$

for all $i \geqslant 0$. We have $\|(v_i \otimes 1)\xi_n\| \leqslant \|\xi_n\| \leqslant 1$; thus,

$$\begin{aligned}\|(v_i \otimes 1)\xi_n\|^2 \geqslant & \|(q_{e_0} \otimes P_i(S+S^*))\xi_n\|^2 \\ & - \|(q_{e_0} \otimes P_i(S+S^*))\xi_n - (v_i \otimes 1)\xi_n\|^2 \\ & - 2\|(q_{e_0} \otimes P_i(S+S^*))\xi_n - (v_i \otimes 1)\xi_n\|.\end{aligned} \tag{8}$$

By Lemma 2.3, there exists an integer $I \in \mathbb{N}$ such that

$$\inf_{t\in[-2;2]} S_I(t) > \frac{2}{\varepsilon}.$$

We have

$$\begin{aligned}\sum_{i=0}^{I} \|(q_{e_0} \otimes P_i(S+S^*))\xi_n\|^2 &= \sum_{i=0} \|(1 \otimes P_i(S+S^*)) \circ (q_{e_0} \otimes 1)\xi_n\|^2 \\ &= \sum_{i=0}^{I} \int_{[-2;2]} \|P_i(t)((q_{e_0} \otimes \Psi)\xi_n)(t)\|^2 d\nu(t) \\ &= \int_{[-2;2]} (\sum_{i=0}^{I} P_i(t)^2) \|((q_{e_0} \otimes \Psi)\xi_n)(t)\|^2 d\nu(t) \\ &\geqslant \frac{2}{\varepsilon}\|(q_{e_0} \otimes \Psi)\xi_n\|^2 = \frac{2}{\varepsilon}\|(q_{e_0} \otimes 1)\xi_n\|^2, \end{aligned} \tag{9}$$

where Ψ is the unitary transformation defined in section 2.3.

By 7, there exists an element of the ultrafilter $E \in \omega$ such that for any $n \in E$ and $i \in \{0, \cdots, I\}$ we have

$$\|((q_{e_0} \otimes P_i(S+S^*)) - (v_i \otimes 1))\xi_n\| < \frac{1}{4}. \tag{10}$$

By Pythagoras theorem and the Inequations 8, 9 and 10 we have

$$\begin{aligned} 1 \geqslant \|\xi_n\|^2 &= \sum_{i\geqslant 0} \|(q_{e_i} \otimes 1)\xi_n\|^2 \\ &\geqslant \sum_{i=0}^{I} \|(q_{e_i} \otimes 1)\xi_n\|^2 = \sum_{i=0}^{I} \|(v_i \otimes 1)\xi_n\|^2 \\ &\geqslant \sum_{i=0}^{I} \|(q_{e_0} \otimes P_i(S+S^*))\xi_n\|^2 - (I+1)(\frac{1}{4^2} + 2\frac{1}{4}) \\ &\geqslant \frac{2(I+1)}{\varepsilon}\|(q_{e_0} \otimes 1)\xi_n\| - (I+1). \end{aligned}$$

This implies

$$\|(q_{e_0} \otimes 1)\xi_n\| \leqslant \varepsilon, \text{ for all } n \in E.$$

We have proved that

$$\lim_{n\to\omega} \|(q_{e_0} \otimes 1)\xi_n\| = 0.$$

Therefore, $\lim_{n\to\omega} \|Q_J(x_n)\| = 0$ and then $xb \perp bx$. Thus, the cup subalgebra $A \subset M$ has the AOP. By Corollary 2.2, $A \subset M$ is a singular MASA. Hence, by Theorem 1.3, the cup subalgebra is maximal amenable.

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